Journal of Statistical Physics, Vol. 42, Nos. 5/6, 1986

Charged Rods in a Periodic Background: A Solvable Model

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Received October 1, 1985

The one-component Coulomb system with logarithmic potential in a periodic background is considered. In one dimension, when the background has the same period as the average interparticle spacing, the system is exactly solvable for three values of the coupling constant. The exact solution exhibits insulating–conducting phase transitions. An heuristic argument is presented which predicts the phase diagram for this system.

KEY WORDS: Kosterlitz–Thouless phase transition; exact solvability; one-component plasma.

1. INTRODUCTION

The two-species classical Coulomb system consisting of positive and negative charged rods is a model system for diverse physical phenomena. In one space dimension, with the constraint that the charged rods alternate in sign, the system models the Kondo effect.[1,2] In two dimensions the system was the model used by Kosterlitz and Thouless in their pioneering work on metastability and phase transitions in two dimensions.[3]

The Coulomb system is prevented from collapsing by including a short-range repulsive potential, stronger than the logarithmic attraction of oppositely charged rods. At high temperatures the system is in a conducting phase—the charges are mobile and will perfectly screen an external charge density in the long wavelength limit. As the temperature is lowered dipole pairs form. When a certain (in general density dependent) critical temperature is reached, the system consists only of dipole pairs. The charges are no longer individually mobile and will not perfectly screen an

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0022-4715/86/0300-0871$05.00/0

external charge density in the long-wavelength limit. The system is in a dielectric (insulating) phase.

Despite the conducting–dielectric phase transition of the Coulomb gas having been extensively studied for over a decade now, some outstanding questions remain. Let us consider one such question.

From both renormalization group arguments[4] and more recently a Monte Carlo simulation[5] it is known that in the limit of zero particle density the Coulomb gas undergoes a dielectric–conducting phase transition at $\Gamma \equiv q^2/k_B T = 4$ (q is the magnitude of the charge of the particles). Now consider a finite system of an equal number of positive and negative particles contained within a disk of radius R. For Γ large enough the system consists of tightly bound dipoles. If a single dipole ionizes into two free charges—one positive and one negative—the cost in energy is

$$\Delta U \sim q^2 \log R \tag{1.1}$$

while the gain in entropy due to an extra mobile charge is

$$-\Delta S \sim -k_B \log R^2 \tag{1.2}$$

R^2 being the order of the allowed volume available to the extra mobile charge. The change in free energy is thus

$$\begin{aligned}\Delta F &\equiv \Delta U - T\Delta S \\ &= (q^2 - 2k_B T) \log R \end{aligned}\tag{1.3}$$

This quantity is negative provided $\Gamma < 2$. Hence on the basis of this argument we would expect the dipoles to ionize at the temperature $\Gamma = 2$, in contradiction to the known result.

What is wrong with the above argument? Correlations have been completely ignored. It is therefore necessary to include the effects of correlations even in the zero density limit. This raises the question of providing a simple heuristic argument which *includes correlations* to predict the transition at $\Gamma = 4$.

In this paper we do not provide such an argument in the two-dimensional domain. However, on the basis of some exact calculations in one dimension, we do show the essential role the correlation functions play in the conducting–insulating transition, and do provide such an argument for the solvable system.

The system we will consider is the one-component Coulomb gas with logarithmic potential in a periodic background. The domain is a line and we impose periodic boundary conditions. The periodic background is chosen to have the same period as the average particle spacing ($1/\mu$). It can have a completely arbitrary origin (i.e., it is not necessarily Coulombic).

This system has the special property of being exactly solvable for three values of the coupling constant: $\Gamma = 1$, 2, and 4. This applies for all periodic potentials (period $1/\mu$) with a Boltzmann factor that has a convergent Fourier series. The free energy and one- and two-particle correlations can all be calculated exactly. From the exact expressions for the two-particle correlations we show explicitly that the system is in a conducting state at $\Gamma = 1$ and insulating at $\Gamma = 2$ and 4, for general periodic backgrounds of period $1/\mu$. However, if the periodic background is such that its first Fourier coefficient vanishes, then the system remains in a conducting phase at both $\Gamma = 2$ and 4.

2. PRELIMINARIES

2.1. Definition of the Model

The pair potential for two unit charged rods interacting in periodic boundary conditions is

$$\phi(\theta, \theta') = -\frac{1}{2}\log\left\{2\left[1-\cos\frac{2\pi}{W}(\theta-\theta')\right]\left(\frac{W}{2\pi}\right)^2\right\} \tag{2.1}$$

where W is the length of the system. On the line of length W suppose there are N mobile charges. As well as the interparticle interaction (2.1) suppose each particle interacts with a periodic background, of period $1/\mu = W/N$. Let the energy of interaction between the particle and background be $U(\theta)$. If necessary, the system is made charge neutral by also imposing a uniform background charge density. Then for both Coulombic and non-Coulombic periodic backgrounds the Hamiltonian for the system is

$$H = -\frac{q^2}{2}\left[-2N\log\left(\frac{W}{2\pi}\right) + N\log N + 2\sum_{1 \leqslant j < k \leqslant N} \log|e^{2\pi i\theta_k/W} - e^{2\pi i\theta_j/W}|\right] + \sum_{k=1}^{N} U(\theta_k) \tag{2.2}$$

Consider the one-body terms in H, due to the particle–periodic background interaction. Since they are periodic of period $1/\mu$ we can write the Boltzmann factor

$$f(\theta_k) \equiv e^{-\beta U(\theta_k)} \tag{2.3}$$

as the Fourier series

$$f(\theta_k)=\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n\mu\theta_k} \tag{2.4}$$

where

$$c_n=c_{-n}=\mu\int_0^{1/\mu} e^{-\beta U(x)}\cos 2\pi\mu n x\,dx \tag{2.5}$$

Thus the Boltzmann factor is

$$e^{-\beta H}=\exp[-4N\Gamma\log(W/2\pi)+2\Gamma N\log N]$$
$$\times\prod_{1\leqslant j<k\leqslant N}|e^{2\pi i\theta_k/W}-e^{2\pi i\theta_j/W}|^{\Gamma}\prod_{k=1}^{N} f(\theta_k) \tag{2.6}$$

where we will use the representation (2.4) of $f(\theta)$.

The free energy per particle and the one- and two-particle correlations in the finite system are thus, respectively,

$$\beta f=-\frac{1}{N}\log[(2\pi/W)^{N\Gamma}N^{\Gamma N/2}(I_{0,N}/N!)] \tag{2.7}$$

$$\rho(\theta_1)=NI_{1,N}/I_{0,N} \tag{2.8}$$

$$\rho(\theta_1,\theta_2)=N(N-1)I_{2,N}/I_{0,N} \tag{2.9}$$

where

$$I_{a,N}\equiv I_{a,N}(f,\mu,\Gamma)$$
$$=\left[\prod_{p=1}^{a} f(\theta_p)\right]\sum_{n_{a+1}=-\infty}^{\infty}\sum_{n_{a+2}=-\infty}^{\infty}\cdots\sum_{n_N=-\infty}^{\infty}\prod_{l=a+1}^{N} c_{n_l}\int_0^W d\theta_l\, e^{2\pi i n_l\mu}D^{\Gamma} \tag{2.10}$$

and

$$D\equiv D(\theta_1,\theta_2,\ldots,\theta_N)=\prod_{1\leqslant j<k\leqslant N}|e^{2\pi i\theta_k/W}-e^{2\pi i\theta_j/W}| \tag{2.11}$$

We can evaluate $I_{a,N}$, $a=0, 1, 2$ for all f and μ at the three values of the coupling constant $\Gamma=1, 2$, and 4. But first we will review some electrostatics, and derive a sum rule which characterizes the phase (conducting or insulating) of the system.

2.2. Conducting and Insulating Phases of Coulomb Systems with the Logarithmic Potential in Strip Geometry

From the static viewpoint, the characteristic feature of the conducting phase of a Coulomb system is perfect screening. The system will respond to an external charge density in the long-wavelength limit so as to exactly cancel that charge density. This feature is characteristic of the conducting phase irrespective of the geometry of the system.

On the other hand, suppose we are in the two-dimensional bulk and in a dielectric phase. In the long-wavelength limit of an external charge density, the system will respond to cancel only the fraction $(1-1/\varepsilon)$ of the external charge density. For the two-dimensional domain these features can be taken as definitions of each phase respectively.

However, we want to consider the dielectric constant of a strip domain. Suppose we cut and place in a vacuum (dielectric constant unity) a strip-shaped slice of two-dimensional dielectric material. Let this material have dielectric constant ε and suppose the strip is of width d and infinite in length with its bottom edge on the X axis of the X–Y plane. The electrostatic potential ϕ at the point (x, y) within the strip created by a unit charge at the point (x', y') also within the strip can be calculated using the method of images. We find

$$\phi(x, y) = -\frac{1}{\varepsilon}\left\{\sum_{n=-\infty}^{\infty} \eta^{2|n|+1} \log[(x-x')^2 + (y+y'-2nd)^2]^{1/2} + \sum_{n=-\infty}^{\infty} \eta^{2|n|} \log[(x-x)^2 + (y-y'-2nd)^2]^{1/2}\right\} \tag{2.12}$$

where

$$\eta = \frac{1-\varepsilon}{1+\varepsilon} \tag{2.13}$$

The large-wavelength behavior of (2.12) is given by the small-k behavior of the Fourier transform

$$\tilde{\phi}(k) = \int_{-\infty}^{\infty} \phi(x, y) e^{ikx}\, dx \tag{2.14}$$

we have

$$\tilde{\phi}(k) \underset{k\to 0}{\sim} \frac{\pi}{|k|} \sum_{m=-\infty}^{\infty} \eta^{|m|} = \frac{\pi}{|k|} \tag{2.15}$$

But this is the small-k behavior of a unit charge in a vacuum

$$\bar{\psi}(k) = \int_{-\infty}^{\infty} \psi(x, y) e^{ikx}\, dx$$

$$\sim \frac{\pi}{|k|} \qquad \text{as} \quad k \to 0 \tag{2.16}$$

where

$$\psi(x, y) = -\tfrac{1}{2} \log(x^2 + y^2) \tag{2.17}$$

Thus from the viewpoint of screening an external charge density in the long wavelength limit, the dielectric strip behaves as a vacuum. Since the line is just the zero strip width limit this remark applies also to a line of dielectric material.

2.3. Sum Rules

The above screening characteristics can be used to derive sum rules which specify the phase (conducting or insulating) in terms of an asymptotic property of the charge–charge correlation. This is done in a standard way using linear response theory.[6,7]

Recall that if the Hamiltonian H of a system can be written

$$H = H_0 + \lambda H_1 \tag{2.18}$$

where λ is small, the linear response relation says that for any observable A,

$$\langle A \rangle_\lambda - \langle A \rangle_0 = \lambda\beta(\langle AH_1 \rangle_0 - \langle A \rangle_0 \langle H_1 \rangle_0) \tag{2.19}$$

Here $\langle \cdot \rangle$ denotes the canonical average, and the subscript 0 indicates the average is taken when $\lambda = 0$. We will take for H_0 the Hamiltonian (2.2).

Suppose the perturbed Hamiltonian is due to an oscillating background density

$$\rho_{\text{ext}}(x, y) = \lambda e^{iky} \delta(x) \tag{2.20}$$

where δ denotes the Dirac delta function. This charge density creates a potential

$$\phi_{\text{ext}}(x, y) = \frac{\pi}{|k|} e^{iky - |kx|} \tag{2.21}$$

which couples to the mobile particles so that

$$\lambda H_1 = \lambda \int C(y)\, \phi(0, y)\, dy \tag{2.22}$$

where

$$C(y) = q \sum_{j=1}^{N} \delta(y - y_j) \tag{2.23}$$

is the microscopic charge density of the mobile particles.

Let us take for the observable A the charge density (2.23), and consider the linear response relation (2.19) in the limit $k \to 0$. In this limit the external charge density tends to the constant $\lambda\delta(x)$, and the induced charge density will have the periodicity of the periodic background. From the characteristics of the conducting and insulating phases noted in the above section, the system will respond to exactly cancel the fraction $(1 - 1/\varepsilon')$ of this charge. Here $\varepsilon' = \infty$ for the conducting phase and $\varepsilon' = 1$ for the insulating phase. Hence the average value of the induced charge density over one period of the background must equal $-\lambda(1 - 1/\varepsilon')$, i.e.,

$$-\lambda(1 - 1/\varepsilon') = \mu \int_0^{1/\mu} dy' [\langle C(y')\rangle_\lambda - \langle C(y')\rangle_0] \qquad \text{as} \quad k \to 0 \tag{2.24}$$

But from (2.19) and (2.21)–(2.23),

$$\langle C\rangle_\lambda - \langle C\rangle_0 = \frac{\pi}{|k|} \beta\lambda \int dy\, e^{iky} C_2^T(y, y') \tag{2.25}$$

where

$$\begin{aligned} C_2^T(y, y') &= \langle C(y)\, C(y')\rangle - \langle C(y)\rangle \langle C(y')\rangle \\ &= q^2[\delta(y - y')\, \rho(y) + \rho_2^T(y, y')] \end{aligned} \tag{2.26}$$

ρ_2^T denoting the truncated two-particle distribution function. Substituting (2.25) in (2.24) we obtain the sum rule

$$\mu \int_0^{1/\mu} dy' \int_{-\infty}^{\infty} dy\, e^{iky} C_2^T(y + y', y') \sim |k|\, (1 - 1/\varepsilon')/\pi\beta \qquad \text{as} \quad k \to 0 \tag{2.27}$$

From (2.26) and Fourier transform theory this is equivalent to saying that the large-y expansion of

$$\mu \int_0^{1/\mu} dy'\, \rho_2^T(y + y', y') \tag{2.28}$$

must contain as its leading order nonoscillatory term

$$-\frac{(1-1/\varepsilon')}{\Gamma\pi^2 y^2} \tag{2.29}$$

Thus in the conducting phase the large-y expansion of (2.28) must contain as its leading order nonoscillatory term

$$-\frac{1}{\Gamma\pi^2 y^2} \tag{2.30}$$

In the insulating phase, when $\varepsilon' = 1$, the leading order nonoscillatory term of (2.28) will by $o(1/y^2)$.

From the exact expressions for ρ_2^T to be calculated below it is this sum rule which determines the phase of the system.

3. THE EXACT RESULTS

We will now proceed to evaluate the integrals $I_{a,N}$ for the three values of the coupling constant $\Gamma = 1$, 2, and 4. We do this in order of difficulty: $\Gamma = 2$, 1, then 4.

3.1. The Coupling Constant $\Gamma = 2$

Free Energy. The Boltzmann factor for the particle–particle interaction is just the product of two Vandermonde determinants.[8] Thus

$$D^2 = \sum_{R=1}^{N!} \sum_{S=1}^{N!} \varepsilon(R)\,\varepsilon(S) \prod_{l=1}^{N} e^{2\pi i\theta_l[R(l)-S(l)]/W} \tag{3.1}$$

where $\varepsilon(R)$ and $\varepsilon(S)$ denotes the signature of the permutations R and S. Inserting this identity in (2.10) with $a=0$ we see that for nonzero contribution to the partition function we require

$$R(l) - S(l) + Nn_l = 0 \qquad \text{for each} \quad l = 1, 2,..., N \tag{3.2}$$

Since $|R(l) - S(l)| < N$ we must have

$$n_l = 0$$

and

$$S(l) = R(l) \qquad \text{for each} \quad l = 1, 2,..., N \tag{3.3}$$

Since $[\varepsilon(R)]^2 = 1$ the resulting expression is independent of the particular permutation R. Thus we have the simple result

$$I_{0,N} = N!(Wc_0)^N \tag{3.4}$$

Inserting (3.4) in (2.7) gives

$$\beta f = -\log[c_0 \mu (2\pi)^2] \tag{3.5}$$

for all N. In particular this is the free energy per particle in the thermodynamic limit $N \to \infty$.

One-Particle Distribution Function. From (2.8) we must calculate $I_{1,N}$. Using the identity (3.1) we see the conditions (3.3) must be satisfied for each $l = 2, 3,..., N$. Note that this implies $R(1) = S(1)$. Thus

$$I_{1,N} = (N-1)! f(\theta_1)(Wc_0)^{N-1} \tag{3.6}$$

so inserting (3.6) and (3.4) in (2.8) we have

$$\rho(\theta_1) = \mu f(\theta_1)/c_0 \tag{3.7}$$

for all N.

Two-Particle Distribution Function. We must now calculate $I_{2,N}$. After using the representation (3.1) we see that for nonzero contribution we must have the conditions (3.3) for each $l = 3, 4,..., N$. If $R(1) = r_1$, $R(2) = r_2$, $1 \leqslant r_1, r_2 \leqslant N$, and $r_1 \neq r_2$ we can have $S(1) = r_1$, $S(2) = r_2$ with $\varepsilon(R)\,\varepsilon(S) = 1$, or $S(1) = r_2$, $S(2) = r_1$ with $\varepsilon(R)\,\varepsilon(S) = -1$. Hence

$$I_{2,N} = (N-2)!\, f(\theta_1) f(\theta_2)(Wc_0)^{N-2} \sum_{r_1=1}^{N} \sum_{\substack{r_2=1 \\ r_1 \neq r_2}}^{N} \{1 - \exp[2\pi i(\theta_1 - \theta_2)(r_1 - r_2)/W]\} \tag{3.8}$$

Inserting (3.8) and (3.4) in (2.9) we obtain $\rho(\theta_1, \theta_2)$ for the finite system. In the limit $N, W \to \infty$ the sums in (3.8) become Riemann integrals. These integrals are easily evaluated to give in the thermodynamic limit

$$\rho(\theta_1, \theta_2) = \rho(\theta_1)\, \rho(\theta_2) - \frac{f(\theta_1) f(\theta_2)}{(c_0)^2} \frac{\sin^2 \pi\mu(\theta_1 - \theta_2)}{\pi^2(\theta_1 - \theta_2)^2} \tag{3.9}$$

3.2. The Coupling Constant $\Gamma = 1$

We must consider the integrals

$$J(v) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \cdots \sum_{n_N=-\infty}^{\infty} \prod_{\alpha=1}^{N} c_{n_\alpha} K(v) \tag{3.10}$$

where

$$K(v)=\int_0^W d\theta_l\,[1+v(\theta_l)]e^{2\pi i n_l/W}D(\theta_1,\ldots,\theta_N) \tag{3.11}$$

The integrals $I_{a,N}$ are related to J by

$$I_{0,N}=J(0) \tag{3.12}$$

$$I_{1,N}=\frac{1}{N}\frac{\delta}{\delta v(\theta_1)}J(v)\bigg|_{v=0} \tag{3.13}$$

$$I_{2,N}=\frac{1}{N(N-1)}\frac{\delta^2 J(v)}{\delta v(\theta_1)\,\delta v(\theta_2)}\bigg|_{v=0} \tag{3.14}$$

Here $\delta/\delta v$ denotes functional differentiation.

Our starting point is the formula[8]

$$K(v)=(-i)^{N/2}\frac{N!}{(N/2)!}\sum_X \varepsilon(P)\prod_{l=1}^{N/2}\psi_{P(2l-1),P(2l)}(v) \tag{3.15}$$

where

$$\begin{aligned}\psi_{P(2l-1),P(2l)}(v)=&\int_0^W d\theta_{2l}\int_0^W d\theta_{2l-1}\,\mathrm{sgn}(\theta_{2l}-\theta_{2l-1})[1+v(\theta_{2l-1})][1+v(\theta_{2l})]\\&\times\exp\left\{2\pi i\frac{\theta_{2l-1}}{W}\left[P(2l-1)-\frac{N+1}{2}+Nn_{2l-1}\right]\right.\\&\left.+2\pi i\frac{\theta_{2l}}{W}\left[P(2l)-\frac{N+1}{2}+Nn_{2l}\right]\right\}\end{aligned} \tag{3.16}$$

and X denotes the sum of all permutations P of $\{1, 2,\ldots, N\}$ such that $P(2l)>P(2l-1)$. To calculate the free energy we set $v=0$. We can then reduce the double integral in (3.16) to a single integral

$$\begin{aligned}&\psi_{P(2l-1),P(2l)}(0)\\&=-\frac{W}{2\pi i}\left[\frac{1}{P(2l)-(N+1)/2+Nn_{2l}}-\frac{1}{P(2l-1)-(N+1)/2+Nn_{2l-1}}\right]\\&\quad\times\int_0^W d\theta\,\exp\{2\pi i\theta[P(2l)+P(2l-1)-N-1+N(n_{2l}+n_{2l-1})]/W\}\end{aligned} \tag{3.17}$$

Since $|P(2l)+P(2l-1)-N-1|<N$, for nonzero contribution to the sum over X in (3.15) we require

$$n_{2l}=-n_{2l-1}$$

and

$$P(2l)+P(2l-1)=N+1 \tag{3.18}$$

for each $l=1, 2,..., N$. Thus we must have

$$P(2l)=N+1-Q(l)$$
$$P(2l-1)=Q(l), \qquad Q(l)\in\{1, 2,..., N/2\} \tag{3.19}$$

where each such permutation has even parity. Therefore

$$J(0)=\frac{W^N N!}{\pi^{N/2}(N/2)!}\sum_{Q=1}^{(N/2)!}\prod_{l=1}^{N/2}\sum_{n=-\infty}^{\infty}\frac{(c_{2n})^2}{(N+1)/2-Q(l)+Nn} \tag{3.20}$$

This expression is independent of the particular permutation Q so we can choose $Q(l)=l$ and multiply by $(N/2)!$ Inserting the resulting expression in (3.12) and (2.7) we obtain the free energy in the finite system. In the thermodynamic limit the sum over l tends to a Riemann integral, and we obtain

$$\beta f= -\frac{1}{2}\log 4\pi-\frac{1}{2}\int_0^1 dx \log\left[\sum_{n=-\infty}^{\infty}\frac{(c_n)^2}{(1-x)/2+n}\right] \tag{3.21}$$

One-Particle Distribution. From the definition of functional differentiation we have

$$\begin{aligned}\frac{\delta}{\delta v(\theta_1)}K(v)\Big|_{v=0} &= (-i)^{N/2}\frac{N!}{(N/2)!}\left(-\frac{W}{\pi i}\right)\sum_X \varepsilon(P)\left(\sum_{j=1}^{N/2}\left[\frac{1}{P(2j)-(N+1)/2+Nn_{2j}}\right.\right.\\ &\quad\left.-\frac{1}{P(2j-1)-(N+1)/2+Nn_{2j-1}}\right]\\ &\quad\left.\times\exp\left\{\frac{2\pi i\theta_1}{W}[P(2j)+P(2j-1)-N-1+N(n_{2j}+n_{2j-1})]\right\}\right)\\ &\quad\times\prod_{\substack{l=1\\ l\neq j}}^{N/2}\psi_{P(2l-1),P(2l)}(0)\end{aligned} \tag{3.22}$$

For nonzero contribution to the sum over X we require the conditions (3.19) for each $l=1, 2,..., N/2$, $l\neq j$. Therefore if $Q(j)=p$

$$\frac{\delta}{\delta v(\theta_1)}J(v)\Big|_{v=0}=J(0)\,f(\theta_1)\frac{2}{W}\sum_{p=1}^{N/2}\frac{\sum_{n=-\infty}^{\infty} c_n\cos 2\pi\mu n\theta_1\Big/\left(\frac{N+1}{2}-p+Nn\right)}{\sum_{n=-\infty}^{\infty}(c_n)^2\Big/\left(\frac{N+1}{2}-p+Nn\right)} \tag{3.23}$$

Substituting (3.23) and (3.13), (3.12) in (2.8) we obtain for the one-particle density in the thermodynamic limit

$$\rho(\theta_1) = \frac{\mu f(\theta_1)}{2} \int_{-1}^{1} dx \frac{g(\theta_1, x)}{\sum_{n=-\infty}^{\infty} (c_n)^2/(n + x/2)} \tag{3.24}$$

where

$$g(\theta, x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i \mu \theta n}/(x/2 + n) \tag{3.25}$$

The Two-Particle Distribution. The intermediate steps in this calculation are cumbersome to write down. Since they require no techniques other than those used above, we will present only the results. With the definition (3.25) of g we find

$$\begin{aligned} \rho(\theta_1, \theta_2) &= \rho(\theta_1)\, \rho(\theta_2) - \frac{f(\theta_1) f(\theta_2) \mu^2}{4} \operatorname{Re} \int_{-1}^{1} dt \int_{-1}^{1} ds \\ &\times \frac{\exp[\pi i \mu(\theta_1 - \theta_2)(t - s)][g(-\theta_1, t)\, g(\theta_2, t) + g(-\theta_1, t)\, g(-\theta_2, s)]}{\left[\sum_{n=-\infty}^{\infty} (c_n)^2/(t/2 + n)\right]\left[\sum_{n=-\infty}^{\infty} (c_n)^2/(s/2 + n)\right]} \\ &+ \pi \mu f(\theta_1) f(\theta_2) \int_0^1 dt \frac{\sin(|\theta_1 - \theta_2| \pi \mu t)}{\sum_{n=-\infty}^{\infty} (c_n)^2/(t/2 + n)} \end{aligned} \tag{3.26}$$

where Re denotes the real part.

3.3. The Coupling $\Gamma = 4$

Free Energy. We use the identity[8]

$$D^4 = \sum_X \varepsilon(P) \prod_{l=1}^{N} [P(2l) - P(2l-1)] \exp\{2\pi i \theta_l [P(2l) + P(2l-1) - 2N - 1]/W\} \tag{3.27}$$

where X denotes the sum over all permutations P of $\{1, 2,..., 2N\}$ such that $P(2l) > P(2l-1)$.

Inserting this expression in the definition (2.10) of $I_{0,N}$ we see that for nonzero contribution to the partition function we require

$$P(2l) + P(2l-1) - 2N - 1 + Nn_l = 0 \tag{3.28}$$

for each $l = 1, 2, \ldots, N$. There are three possibilities:

$$P(2l) + P(2l-1) - 2N - 1 = \begin{cases} -N, & n_l = 1 \quad [\text{type } (-)] \\ 0, & n_l = 0 \quad [\text{type } (0)] \\ N, & n_l = -1 \quad [\text{type } (+)] \end{cases} \tag{3.29}$$

We will use the symbolic notation $(-)$, (0), and $(+)$ as indicated in (3.29). For such choices of permutations we have

$$I_{0,N} = W^N \sum_X \varepsilon(P) \prod_{l=1}^{N} c_{n_l}[P(2l) - P(2l-1)] \tag{3.30}$$

Our method is to consider the value of (3.30) firstly with type (0) for each l, then consider in turn the effect of one, two,... type $(-)$, $(+)$. To calculate the contribution to $I_{0,N}$ (A_0 say) from all permutations of type (0), we note such permutations must be of the form

$$\begin{aligned} P(2l) &= 2N + 1 - Q(l) \\ P(2l-1) &= Q(l) \end{aligned} \tag{3.31}$$

for each $l = 1, 2, \ldots, N$ where $Q(l)$ is a permutation on $\{1, 2, \ldots, N\}$. Thus since all such permutations have even parity

$$\begin{aligned} A_0 &= (Wc_0)^N \sum_{Q=1}^{N!} \prod_{l=1}^{N} [2N + 1 - 2Q(l)] \\ &= (Wc_0)^N N! \prod_{l=1}^{N} (2N + 1 - 2l) \end{aligned} \tag{3.32}$$

We note that it is not possible to have a type $(-)$ contribution to (3.30) without having a type $(+)$ (and vice versa) since it would not permit the pairing (3.31) for the remaining type (0)'s.

So next we consider the contribution to (3.30) (to be denoted A_1) from permutations of the form one type $(-)$, one type $(+)$, and $(N-2)$ type (0)'s. Firstly we note that there are $N!/(N-2)!$ different choices amongst the integrations for this class of nonzero contribution. If we multiply by this factor, we can suppose the type $(-)$ occurs for $l = 1$ and the type $(+)$ for $l = 2$. Thus for $l = 1$ we can have

$$\begin{aligned} P(1) &= r \\ P(2) &= N + 1 - r \end{aligned} \tag{3.33}$$

where $r \in \{1, 2, \dots, N/2\}$ (it is thus convenient to choose N even). This immediately implies that

$$\begin{aligned} P(3) &= N + r \\ P(4) &= 2N + 1 - r \end{aligned} \tag{3.34}$$

For $l = 3, 4, \dots, N$ we must have (3.31) where $Q \in \{1, 2, \dots, N\} - \{r, N+1-r\}$. All such permutations have even parity so we have

$$A_1 = (Wc_0)^N N! \left(\frac{c_1}{c_0}\right)^2 \prod_{l=1}^{N} (2N+1-2l) \sum_{r=1}^{N/2} a(r) \tag{3.35}$$

where

$$a(r) = \frac{(N+1-2r)^2}{(2N+1-2r)(2r-1)} \tag{3.36}$$

Proceeding similarly, if we denote by A_n the value of (3.30) from all permutations of the form n type $(-)$, n type $(+)$, and $N-2n$ type (0) $(N-2n \geqslant 0)$ we find

$$A_n = (Wc_0)^N N! \left(\frac{c_1}{c_0}\right)^{2n} \prod_{l=1}^{N} (2N+1-2l) \sum_{1 \leqslant r_1 < r_2 \cdots \leqslant N/2} \prod_{l=1}^{n} a(r_l) \tag{3.37}$$

Hence, since A_n is just the coefficient in the power series expansion of a product we have the formula

$$\begin{aligned} I_{0,N} &= \sum_{n=0}^{N/2} A_n \\ &= (Wc_0)^N \frac{(2N)!}{2^N} \prod_{l=1}^{N/2} \left[1 + \left(\frac{c_1}{c_0}\right)^2 a(l)\right] \end{aligned} \tag{3.38}$$

Substituting (3.38) in (2.7) we obtain the expression for βf in the finite system. The thermodynamic limit can be computed using Stirling's formula, and noting that the series obtained after taking the logarithm of the product in (3.38) converges to a Riemann integral. This integral can be evaluated to give

$$\begin{aligned} \beta f = &-\log[c_0 \mu^3 (2\pi)^5/\pi] + 1 \\ &- \frac{1}{2}\left\{\left[1 - \left(\frac{1}{1-\xi}\right)^{1/2}\right] \log \xi + 2\left(\frac{1}{1-\xi}\right)^{1/2} \log[(1+\sqrt{1-\xi})/2]\right\} \end{aligned} \tag{3.39}$$

where

$$\xi = (c_1/c_0)^2 \tag{3.40}$$

One-Particle Distribution. For the choices of the permutation P given by (3.29) we have

$$I_{1,N}=W^{N-1}f(\theta_1)\sum_X \varepsilon(P)\left(\prod_{l=2}^{N} c_{n_l}\right)\exp\{2\pi i\theta_1[P(2)+P(1)-2N-1]/W\}$$
$$\times\prod_{l=1}^{N}[P(2l)-P(2l-1)] \tag{3.41}$$

To evaluate $I_{1,N}$ we consider in turn the three cases when $P(2)+P(1)-2N-1$ is of type (0), $(-)$, and $(+)$. If we denote the sum contribution of each case by B_0, B_-, and B_+, respectively, then proceeding as above we find

$$B_0=(Wc_0)^{N-1}(N-1)!\prod_{l=1}^{N}(2N+1-2l)$$
$$\times\left[N+\sum_{k=1}^{N/2}(N-2k)\left(\frac{c_1}{c_0}\right)^2\sum_{1\leqslant r_1<\cdots<r_k\leqslant N/2}\prod_{j=1}^{k}a(r_k)\right] \tag{3.42}$$

$$B_-+B_+=(Wc_0)^{N-1}(N-1)!\left[\prod_{l=1}^{N}(2N+1-2l)\right]2\left(\frac{c_1}{c_0}\right)\cos 2\pi\theta_1/W$$
$$\times\sum_{k=1}^{N/2}k\sum_{1\leqslant r_1<\cdots<r_k\leqslant N/2}\prod_{j=1}^{k}a(r_k) \tag{3.43}$$

The term on the second line in (3.42) is just the derivative with respect to x of the product

$$\prod_{l=1}^{N/2}\left[x^2+\left(\frac{c_1}{c_0}\right)^2 a(l)\right] \tag{3.44}$$

evaluated at $x=1$. The term on the second line of (3.43) is the derivative with respect to x of the product

$$\prod_{l=1}^{N/2}\left[1+x\left(\frac{c_1}{c_0}\right)^2 a(l)\right] \tag{3.45}$$

evaluated at $x=1$. Hence

$$I_{1,N}=\frac{2f(\theta_1)I_{0,N}}{Wc_0}\left[\sum_{k=1}^{N/2}\frac{1}{1+(c_1/c_0)^2a(k)}\right.$$
$$\left.+\left(\frac{c_1}{c_0}\right)\cos 2\pi\mu\theta_1\sum_{k=1}^{N/2}\frac{a(k)}{1+(c_1/c_0)^2a(k)}\right] \tag{3.46}$$

Substituting (3.46) in (2.8) and taking the thermodynamic limit we have

$$\rho(\theta_1)=\frac{\mu f(\theta_1)}{c_0}\left[\int_0^1 dt\,\frac{1}{1+\xi\alpha(t)}+\left(\frac{c_1}{c_0}\right)\cos 2\pi\mu\theta_1\int_0^1 dt\,\frac{\alpha(t)}{1+\xi\alpha(t)}\right] \tag{3.47}$$

where

$$\alpha(t)=\frac{(1-t)^2}{t(2-t)} \tag{3.48}$$

and ξ is defined by (3.40). These integrals can be evaluated to give

$$\rho(\theta_1)=\frac{\mu f(\theta_1)}{c_0}\left[1+\left(\frac{c_1}{c_0}\right)\left(\cos 2\pi\mu\theta_1-\left(\frac{c_1}{c_0}\right)\right)\left(-\frac{1}{2}(1-\xi)^{-3/2}\log\xi\right.\right.$$
$$\left.\left.+\frac{1}{\xi}[1-(1-\xi)^{-1/2}]\right)+(1-\xi)^{-3/2}\log\left((1+\sqrt{1-\xi})/2-\frac{(1-\xi)^{-3/2}}{1+\sqrt{1-\xi}}\right)\right] \tag{3.49}$$

Two-Particle Distribution. The problem here is to evaluate

$$I_{2,N}=W^{N-2}f(\theta_1)\,f(\theta_2)\sum_X\varepsilon(P)\prod_{l=3}^{N}c_{n_l}\prod_{l=1}^{N}[P(2l)-P(2l-1)]$$
$$\times\exp\{2\pi i\theta_1[P(2)+P(1)-2N-1]/W$$
$$+2\pi i\theta_2[P(4)+P(3)-2N-1]/W\} \tag{3.50}$$

where the permutations $P(5)$, $P(6)$,..., $P(2N)$ are given by (3.29). The remaining permutations can be categorized as belonging to one of six classes: $(0,0)$, $(0,+)$, $(0,-)$, $(+,+)$, $(+,-)$, $(-,-)$. Thus, for example, if $P(1)$,..., $P(4)$ belong to the class $(0,-)$ then subject to the constraints $P(2)>P(1)$, $P(4)>P(3)$ we can have $P(1),\ldots,P(4)\in\{2N+1-p,\ p,\ N+1-r,\ r\}$, where $1\leqslant p\leqslant N$ and $1\leqslant r\leqslant N/2$.

We now proceed as in the calculation for the free energy. The details are long but straightforward. We find

$$\frac{\rho^T(\theta_1,\theta_2)}{(1/c_0)^2 f(\theta_1)\,f(\theta_2)}=-\frac{\mu^2}{4}\,\mathrm{Re}\int_0^1 dt\int_0^1 ds$$
$$\times\frac{\exp[2\pi i\mu(\theta_1-\theta_2)(t-s)](2-t-s)^2-\exp[2\pi i\mu(\theta_1-\theta_2)(t+s-2)](t-s)^2}{(1-t)(1-s)[1+\xi\alpha(2t)][1+\xi\alpha(2s)]}$$
$$-\left(\frac{c_1}{c_0}\right)\frac{\mu^2}{2}\,\mathrm{Re}\int_0^1 dt\int_0^1 ds\,\frac{\alpha(t)}{(1-t)(1-s)[1+\xi\alpha(t)][1+\xi\alpha(2s)]}$$

$$\times\Big\{(\exp(2\pi i\mu\theta_1)\exp[2\pi i\mu(\theta_1-\theta_2)(t/2-s)]$$

$$+\exp(2\pi i\mu\theta_2)\exp[-2\pi i\mu(\theta_1-\theta_2)(t/2-s)])\left(1-\frac{t}{2}-s\right)\left(2-\frac{t}{2}-s\right)$$

$$+\{\exp(2\pi i\mu\theta_1)\exp[2\pi i\mu(\theta_1-\theta_2)(1-t/2-s)]$$

$$+\exp(2\pi i\mu\theta_2)\exp[-2\pi i\mu(\theta_1-\theta_2)(1-t/2-s)]\}\left(s-\frac{t}{2}\right)\left(1+\frac{t}{2}-s\right)\Big\}$$

$$+\left(\frac{c_1}{c_0}\right)^2\frac{\mu^2}{2}\operatorname{Re}\int_0^1 dt\int_0^1 ds\,\frac{\alpha(t)\,\alpha(s)}{(1-t)(1-s)[1+\xi\alpha(t)][1+\xi\alpha(s)]}$$

$$\times\Big\{-\left[1-\frac{(t-s)^2}{4}\right]\exp\{2\pi i\mu(\theta_1-\theta_2)[(t+s)/2-1]\}$$

$$+\left(\frac{t+s}{2}\right)\left(2-\frac{t+s}{2}\right)\exp[\pi i\mu(\theta_1-\theta_2)(t-s)]$$

$$-\left(1-\frac{t+s}{2}\right)^2\exp[2\pi i\mu(\theta_1+\theta_2)]\exp[\pi i\mu(\theta_1-\theta_2)(t-s)]$$

$$+\frac{1}{8}(t-s)^2\{\exp(4\pi i\mu\theta_2)\exp[\pi i(\theta_1-\theta_2)(r+s)]$$

$$+\exp(4\pi i\mu\theta_1)\exp[-\pi i(\theta_1-\theta_2)(r+s)\}\Big\} \tag{3.51}$$

3.4. A Mathematical Conjecture

Before we go on to discuss these mathematical results from the physical viewpoint, let us first consider the possibility of generalizing them. The simplest case to consider is that for which the only nonzero Fourier coefficients of the Boltzmann factor of the periodic background are c_0 and c_1.

Let us denote the value of the integral $I_{0,n}$ in this case by $J_N(\Gamma)$. Then from (3.20), (3.4), and (3.38)

$$J_N(1)=(Wc_0)^N K_{1,N}\prod_{l=1}^{N/2}\left[1-\frac{1}{2}\left(\frac{c_1}{c_2}\right)^2\frac{(N+1-2l)^2}{(3N/2-l+1/2)(N/2+l-1/2)}\right] \tag{3.52}$$

$$J_N(2)=(Wc_0)^N K_{2,N} \tag{3.53}$$

$$J_N(4)=(Wc_0)^N K_{4,N}\prod_{l=1}^{N/2}\left[1+\left(\frac{c_1}{c_0}\right)^2\frac{(N+1-2l)^2}{(2N+1-2l)(2l-1)}\right] \tag{3.54}$$

where

$$K_{\Gamma,N}=\frac{(\Gamma N/2)!}{[(\Gamma/2)!]^N} \tag{3.55}$$

On the basis of these results we conjecture that for N even and Γ a positive integer

$$J_N(\Gamma)=(Wc_0)^N K_{\Gamma,N}\prod_{l=1}^{N/2}\left[1+\left(\frac{c_1}{c_0}\right)^2 a(\Gamma,N,l)\right] \tag{3.56}$$

Note that when $c_1=0$ the integral $J_N(\Gamma)$ is just the partition function for the one-component log-gas, and the value given by (3.55) is known to be correct.[8] Furthermore from the definition of $J_N(\Gamma)$ we see that for Γ and N even it is a polynomial in $(c_1/c_0)^2$ of order $N/2$.

Our hope is that it will be possible to evaluate $a(\Gamma, N, l)$ for all positive integers Γ, and thus by Carlson's theorem[8] for all Γ.

4. THE PHASE OF THE EXACT ISOTHERMS

The phase of the system (conducting or insulating) can be determined from the sum rules of Section 2.3. But before we do this, as a test of the accuracy of our results, we will check the perfect screening sum rule.[9] This sum rule is necessary for Coulomb systems to be stable. It says

$$\int_{-\infty}^{\infty} dy\, C_2^T(y,y')=0 \tag{4.1}$$

or equivalently using (2.26)

$$-\rho(y')=\int_{-\infty}^{\infty} dy\, \rho^T(y,y') \tag{4.2}$$

Using the exact results (3.7), (3.9), (3.24), (3.26), (3.47), and (3.51) we can check that (4.2) holds in each case.

4.1. Large-y Expansions of the Truncated Two-Body Distributions

The sum rules of Section 2.3 require the large-y expansions of the quantity

$$S(y)=\mu\int_0^{1/\mu} dy'\, \rho^T(y+y',y') \tag{4.3}$$

From the exact expressions (3.26), (3.9), and (3.51) of the correlations we deduce the following.

$\Gamma=1$:

$$S(y)\sim -\frac{1}{(\pi y)^2}-\frac{1}{(\pi y)^2}$$
$$\times\left[\sum_{n=-\infty}^{\infty}\frac{(c_n)^2}{n+1/2}\right]^{-2}\sum_{\substack{n_1=-\infty\\ n_1\neq 0}}^{\infty}\sum_{n_2=-\infty}^{\infty}\sum_{n_3=-\infty}^{\infty}\frac{c_{n_2+n_1}c_{n_3+n_1}c_{n_2}c_{n_3}\cos 2\pi\mu y n_1}{(1/2+n_3+n_1)(1/2+n_2)} \tag{4.4}$$

$\Gamma=2$:

$$S(y)\sim -\frac{[1-(c_1/c_0)^2]}{2(\pi y)^2}-\frac{1}{(c_0\pi y)^2}$$
$$\times\sum_{m=1}^{\infty}\left[(c_m)^2-\frac{1}{2}(c_{m-1})^2-\frac{1}{2}(c_{m+1})^2\right]\cos 2\pi\mu y m \tag{4.5}$$

$\Gamma=4$:

$$S(y)\sim\frac{1}{2(c_1\pi y)^2}\sum_{m=1}^{\infty}(c_{m+1}-c_{m-1})^2\cos 2\pi\mu y m$$
$$+\frac{1}{y^3}\sum_{n=0}^{\infty}a_n\sin 2\pi(n+1/2)\,\mu y+\frac{b_0}{y^4},\qquad c_1\neq 0 \tag{4.6}$$

$$S(y)\sim\frac{1}{4(c_0)^2y}\sum_{n=1}^{\infty}[(c_{n+1})^2+(c_{n-1})^2]\cos 2\pi n\mu y$$
$$-\frac{1}{4(\pi y)^2}-\frac{1}{2(\pi c_0 y)^2}\sum_{n=1}^{\infty}(c_n)^2\cos 2\pi n\mu y$$
$$-\frac{1}{8\pi(yc_0)^2}\sum_{n=1}^{\infty}[(c_{n-1})^2-(c_{n+1})^2]\sin 2\pi n\mu y,\qquad c_1=0 \tag{4.7}$$

In (4.6) b_0 and the a_n are functions of the c_n, and we have not written down the oscillatory terms of order $1/y^4$.

Thus from the sum rules of Section 2.3 we see that at $\Gamma=1$ the system is in a conducting phase for all periodic backgrounds of period $1/\mu$. At $\Gamma=2$, if $c_1\neq 0$ the coefficient of the $1/y^2$ term in (4.5) does not fit in our classification of phases, so it is neither conducting nor dielectric. But if $c_1\neq 0$ at $\Gamma=4$, there is no nonoscillatory term of order $1/y^2$, so from Section 2.3 the system is in a dielectric phase. If $c_1 = 0$ at $\Gamma = 2$ or $\Gamma = 4$, again the system is in a conducting phase.

The relevant parameters with regard to phase transitions in this system are thus Γ and c_1. For a given periodic background with $c_1 \neq 0$, varying Γ from $\Gamma = 1$ to $\Gamma = 4$ changes the phase from conducting to insulating. But since we have the exact solution for three discrete values of Γ only, we have no knowledge of the nature of the phase transition.

In the nonconducting phases $\Gamma = 2$ and $\Gamma = 4$ varying c_1 from positive to negative values (or vice versa) induces a transition to a conducting phase at $c_1 = 0$. From the exact expressions for the free energy (3.5), one-particle correlation (3.7), and two-particle correlation (3.9) this transition is not accompanied by any singularities at $\Gamma = 2$. But in the vicinity of $c_1 = 0$ at $\Gamma = 4$, from (3.39), the singular part of βf behaves as

$$\beta f_{\text{sing}} \sim \tfrac{1}{2}(c_1/c_0)^2 \log|c_1| \tag{4.8}$$

while from (3.49) the singular part of $\rho(\theta)$ behaves as

$$\rho_{\text{sing}}(\theta) \sim -\frac{\mu f(\theta)}{(c_0)^2} (\cos 2\pi\mu\theta) c_1 \log|c_1| \tag{4.9}$$

4.2. Interpretations of the Phases of the Exact Isotherms

We seek a better understanding of the Γ, c_1 behavior observed on the exact isotherms.

As an approximation let us consider the system as consisting of two independent effects: the particles interacting amongst themselves and the particles interacting with the background. Then the truncated two-body distribution can be written

$$\rho^T(\theta, \theta') = A f(\theta) f(\theta') \rho^{*T}(\theta - \theta') \tag{4.10}$$

where A is a normalization constant and $\rho^{*T}(\theta - \theta')$ denotes the truncated two-body distribution of the system in a uniform background (i.e., all the c_n except c_0 equal to zero).

We have an asymptotic formula due to Haldane,[10,11] which says

$$\rho^{*T}(y) \sim -\frac{1}{(\pi y)^2 \Gamma} + \sum_{n=1}^{\infty} a_n \frac{\cos 2\pi n\mu y}{y^{4n^2/\Gamma}} \tag{4.11}$$

where the a_n's are dependent of Γ. Note from the coefficient of the $1/y^2$

term that this homogeneous system is a conductor for all Γ. From Haldane's formula the leading order behavior is

$$\rho^{*T}(y) \sim \frac{-1}{(\pi y)^2 \Gamma}, \quad \Gamma < 2; \qquad \frac{a_1 \cos 2\pi\mu y}{y^{4/\Gamma}}, \quad \Gamma > 2 \tag{4.12}$$

Substituting (4.12) in (4.10) we thus have the leading order nonoscillatory behavior of $S(y)$:

$$S(y) \sim -\frac{A(c_0)^2}{(\pi y)^2 \Gamma}, \quad \Gamma < 2; \qquad \frac{Aa_1(c_1)^2}{y^{4/\Gamma}}, \quad \Gamma > 2 \tag{4.13}$$

Hence if $\Gamma < 2$, by a proper choice of the normalization constant $[A = 1/(c_0)^2]$ we have the correct $1/y^2$ behavior for a conducting phase. However, if $\Gamma > 2$ and $c_1 \neq 0$ this approximation gives a leading order nonoscillatory behavior slower than $1/y^2$. Since we expect the state to be either conducting or insulating, from the sum rules of Section 2.3 such behavior is not possible. We interpret this as indicating that the system has undergone a phase transition at $\Gamma = 2$.

Within the framework of the approximation (4.10), the mechanism for the phase transition is constructive interference between the dominant oscillatory term of the correlation ρ^{*T} for $\Gamma > 2$, and the corresponding

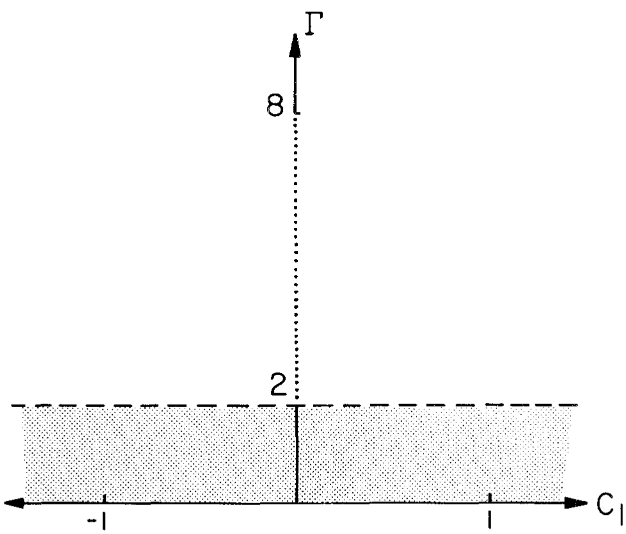

Fig. 1. Conjectured phase diagram in the c_1–Γ plane. The shaded region is conducting. Note that the region of the Γ axis $2 < \Gamma < 8$ is conducting.

oscillatory term in $f(\theta_1)$, which is $c_1 \cos 2\pi\mu y$. If $c_1 = 0$ then there is no constructive interference with the leading order term in $\rho^{*\Gamma}$. Hence, if for a given temperature $\Gamma \geqslant 2$ the next oscillatory term $a_2 \cos 4\pi y/y^{16/\Gamma}$ falls off faster than the nonoscillatory term $-1/\Gamma(\theta y)^2$ (i.e., $\Gamma < 8$) and $c_1 = 0$, then the system will be in a conducting state. This is precisely what we observe at $\Gamma = 2$ and 4 in the exact solution.

On the basis of this approximation we thus predict the phase diagram in the c_1–Γ plane as given by Fig. 1. Note that for the region $\Gamma > 8$, $c_1 = 0$ to be insulating we require $c_2 \neq 0$.

We noted in Section 4.1 that from the static viewpoint the isotherm $\Gamma = 2$, $c_1 \neq 0$ is neither conducting nor insulating. This isotherm, which we believe to be the phase boundary, is such that for the uniform background system, the oscillatory term and the $1/y^2$ nonoscillatory term are of the same order. Thus within the framework of the approximation (4.10) [which from (3.39) is exact at $\Gamma = 2$, !] there is a balance between the constructive interference (insulating behavior) and $1/y^2$ conducting behavior.

4.3. Fixed Positive Charges as Background

We have seen that the phase diagram of the model is dependent only on the first Fourier coefficient of the Boltzmann factor of the particle–background interaction. However, the details of the periodic background do have some distinguishing features.

Consider the case in which the periodic background is due to fixed positive charges of strength q. To prevent the mobile negative charges from collapsing onto the positive charges let there be an impenetrable barrier of total length ε, centered on each positive charge ($\varepsilon < 1/\mu$). The Fourier coefficients of the Boltzmann factor are then

$$c_n = \int_{\mu\varepsilon/2}^{1-\mu\varepsilon/2} \frac{\cos 2\pi n x}{\sin^\Gamma x\pi}\, dx \tag{4.14}$$

Note that for all $\Gamma > 0$, by varying $\mu\varepsilon$, c_n can taken both positive and negative values. Thus for $2 < \Gamma < 8$ the system undergoes an insulating–conducting phase transition at $c_1 = 0$.

Our concern with this explicit case will be the studying of the free energy in the limit $\mu \to 0$. In this limit we would expect to be able to distinguish the conducting and insulating phases. The insulating phase consists of fixed dipoles—each mobile negative charge will pair up with a fixed positive charge. In the limit $\mu \to 0$ the free energy will be only due to the internal energy, and will be finite. In the conducting phase the mobile negative charges are not bound to the positive charges. Thus in the limit

$\mu \to 0$ both the internal energy and entropy will diverge, so we would expect the free energy to diverge. Let us test these expectations on our exact results.

To obtain the $\mu \to 0$ limit of the expression (3.21) for the free energy at $\Gamma = 1$, we use the readily verifiable identities

$$\begin{aligned} S_1 &\equiv \sum_{n=-\infty}^{\infty} \frac{c_n e^{2\pi i \mu n t}}{n+\alpha} \\ &= \mu \left[\frac{2\pi i e^{-2\pi i \mu t \alpha}}{e^{2\pi i \alpha} - 1} \int_0^{1/\mu} dx\, f(x) e^{2\pi i x \mu \alpha} \right. \\ &\quad \left. + 2\pi i e^{-2\pi i \mu t \alpha} \int_0^t dx\, f(x) e^{2\pi i x \mu \alpha} \right] \end{aligned} \tag{4.15}$$

$$\begin{aligned} S_2 &\equiv \sum_{n=-\infty}^{\infty} \frac{(c_n)^2}{n+\alpha} \\ &= \mu \int_0^{1/\mu} dt\, f(t) S_1 \end{aligned} \tag{4.16}$$

Using integration by parts we can then show

$$S_2 \sim \frac{4}{\pi} \cot \pi\alpha (\log \mu\varepsilon)^2 \tag{4.17}$$

and thus from (3.21), at $\Gamma = 1$

$$\beta f \sim -\log(\log \mu\varepsilon) \tag{4.18}$$

Integration by parts of (4.14) gives immediately the small-μ behavior of the c_n for $\Gamma > 1$. We then have from (3.5) that at $\Gamma = 2$

$$\beta f \to -\log(8\varepsilon^{-1}) \qquad \text{as} \quad \mu \to 0 \tag{4.19}$$

and at $\Gamma = 4$

$$\beta f \to -\log(64\varepsilon^{-3}/3) + 1 \qquad \text{as} \quad \mu \to 0 \tag{4.20}$$

Thus, as expected, the free energy diverges in the limit $\mu \to 0$ for the conducting regime $\Gamma = 1$, but is finite in the nonconducting regimes $\Gamma = 2$ and 4.

5. CONCLUSION

The one-component Coulomb system of charged rods confined to one-dimension in a periodic background (period $1/\mu$) exhibits conducting–insulating phase transitions. These transitions occur irrespective of the source of the periodicity of the background. They are dependent only on the temperature and the first Fourier coefficient of the Boltzmann factor of the periodic potential. A mechanism for this transition is given in terms of an approximation whereby the system consists of two independent effects: the particle–particle and particle–background correlations. Within the framework of this approximation the phase transition occurs when oscillatory terms dominate the particle–particle correlation, and interfere constructively with the oscillatory term of the same period in the Boltzmann factor for the particle–background interaction. This gives a heuristic argument predicting the phase diagram of this system.

It remains as an outstanding problem to do the same for corresponding two-dimensional systems. If we consider the one-component charged rods system in a two-dimensional domain, what class of periodic potentials will produce a conducting–dielectric phase transition? What is the underlying mechanism of this transition (especially if the periodic potential is of non-Coulombic origin)? In the low density limit, is the conducting–dielectric transition temperature always $\Gamma = 4$?

ACKNOWLEDGMENTS

I thank B. Jancovici for his hospitality at Orsay where most of this work was done, and A. Alastuey for his hospitality and interest in this work. Also I thank Ph. Choquard for his hospitality at Lausanne and discussions which led to the formulation of this model.

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